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Stable Observers for Motion Estimation of Rigid Body Systems using Lie Group Methods

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Outline

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Attitude Representations

- Rigid body attitude (orientation) is represented *globally and uniquely* by the set of rotation matrices, which are orthogonal matrices with determinant +1.
  - This set is denoted SO(3), and it is a Lie group (an algebraic group that is also a differentiable manifold) of dimension three.
- Several other attitude representations exist, most common being the 3-2-1 Euler angles, unit quaternions, and (modified) Rodrigues parameters
  - Three-parameter representations, like the Euler angle sets and the (modified) Rodrigures parameters, suffer from kinematic/geometric singularities, commonly termed "gimbal lock." This is because they are *local coordinate* representations of attitude.
  - The Lie group of unit quaternions $\mathbb{S}^3$ *double covers* SO(3), ($\mathbb{S}^3/\mathbb{Z}_2 \simeq \mathrm{SO}(3)$). Therefore although it can represent rotations globally, every rotation matrix (attitude) can be represented by exactly two sets of unit quaternions.

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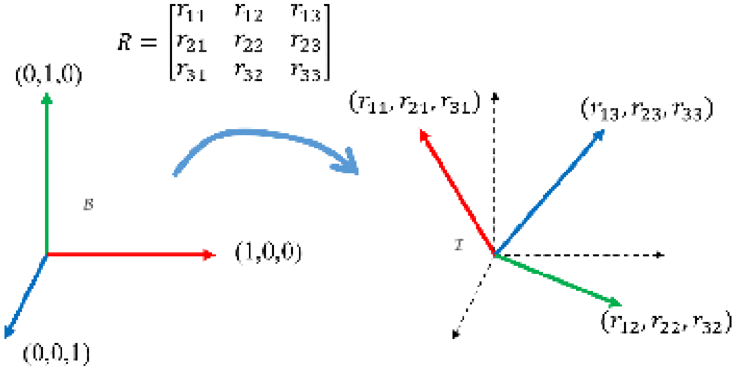
SO(3) and $\mathbb{S}^3$

SO(3) is the natural representation set (global and unique) for rigid body attitude. Due to the double covering of SO(3) by $\mathbb{S}^3$, any continuous observer or feedback controller design using unit quaternions could be unstable due to a phenomenon called *unwinding* (Bhat and Bernstein (2000); Chaturvedi, Sanyal and McClamroch (2011).

Attitude Representation on SO(3)



A rotation matrix from coordinate frame $\mathcal{B}$ to coordinate frame $\mathcal{I}$ represents a 3D rotation as a transformation. Frame $\mathcal{B}$ could be a rigid body-fixed frame and frame $\mathcal{I}$ an inertial (spatial) coordinate frame, in which case the rotation matrix $R \in \mathrm{SO}(3)$ represents the attitude of the rigid body. The *direction cosine matrix* $C = R^{\mathrm{T}}$ would be the inverse transformation, and can also be used as an attitude representation.

Pose Representations

- Rigid body pose (position and orientation) is represented *globally and uniquely* by the union of the set of rotation matrices and three-dimensional Euclidean vector space.
  - This set is denoted SE(3), and it is a Lie group of dimension six.
  - SE(3) is a *semi-direct product* of SO(3) and $\mathbb{R}^3$, denoted $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$.
- Other pose representations consist of product sets of three-dimensional vector space with other attitude representation sets, or the set of *double quaternions*
  - These representations have the same limitations as the representations of the attitude part, i.e., singularities for three parameter attitude representations and lack of uniqueness leading to possible unwinding behavior for $\mathbb{R}^3 \times \mathbb{S}^3$ and the set of double quaternions.

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Semi-direct Product

A semi-direct product of two groups H and N is a product of these groups such that N is a normal subgroup of G, i.e., if $n \in N$ and $g \in G$ then $g' = ngn^{-1} \in G$. G is said to be a semi-direct product of H acting on N, denoted $G = H \ltimes N$. For $\mathrm{SE}(3)$, the normal subgroup is $\mathbb{R}^3$. Elements of $\mathrm{SE}(3)$ are obtained as "rotations acting on translations," as explained in the following slide.

Pose Representation on SE(3)

- The 3D pose of a rigid body can be represented on SE(3) using a "frame transformation" matrix:

$$g = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3) \subset \mathbb{R}^{4\times 4}, \text{ where } R \in \mathrm{SO}(3),\ b \in \mathbb{R}^3.$$

  - The 0 in the bottom left of the block matrix g represents a 1×3 row matrix of zeros.
  - Can be verified that if $g_1, g_2 \in \mathrm{SE}(3)$, then $g_1 g_2 \in \mathrm{SE}(3)$ where the group operation is the matrix product.
- To show that $\mathbb{R}^3$ is the normal subgroup of the semi-direct product SE(3), note that if $p \in \mathbb{R}^3$ and $I \in \mathrm{SO}(3)$ denotes the (3×3) identity matrix, then

$$n = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} \simeq p \in \mathbb{R}^3 \subset \mathrm{SE}(3) \text{ and } gng^{-1} = \begin{bmatrix} I & Rp \\ 0 & 1 \end{bmatrix} \simeq Rp \in \mathbb{R}^3.$$

- The product gn shows the action of SE(3) on $\mathbb{R}^3$ is a rotation followed by a translation, $p \to Rp + b$.

Attitude and Pose Kinematics

- Attitude kinematics relates the time derivative of the rotation matrix to the angular velocity vector $\Omega \in \mathbb{R}^3$ according to

$$\dot{R} = R\Omega^\times, \text{ where } (\cdot)^\times : \mathbb{R}^3 \to \mathfrak{so}(3)$$

  is the skew-symmetric cross product operator map, and $\mathfrak{so}(3)$ is the vector space of 3×3 skew-symmetric matrices, identified with the *Lie algebra* of SO(3).
- Pose kinematics relates the time derivative of the pose $g \in \mathrm{SE}(3)$ to the body velocities according to

$$\dot{g} = g\xi^\vee, \text{ where } \xi = \begin{bmatrix} \Omega \\ \nu \end{bmatrix} \in \mathbb{R}^6 \text{ and } \xi^\vee = \begin{bmatrix} \Omega^\times & \nu \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3).$$

  - $\mathfrak{se}(3)$ denotes the six-dimensional Lie algebra (tangent space at identity) of the six-dimensional Lie group of rigid body pose, SE(3).

Attitude Determination on SO(3)

Attitude determination from vector measurements

- Attitude can be determined uniquely from two or more non-collinear vector measurements made in a body-fixed frame.
- Consider a matrix of $k \geq 2$ vectors, measured and expressed in body frame $\mathcal{B}$:

$$U^m = [u_1^m \; u_2^m \; u_1^m \times u_2^m] \in \mathbb{R}^{3\times 3} \text{ when } k = 2, \text{ and}$$
$$U^m = [u_1^m \; u_2^m \; ... u_k^m] \in \mathbb{R}^{3\times k} \text{ when } k > 2.$$

- The corresponding vectors are known and expressed in inertial frame $\mathcal{I}$ as:

$$E = [e_1 \; e_2 \; e_1 \times e_2] \text{ when } k = 2, \text{ and}$$
$$E = [e_1 \; e_2 \; ... e_k] \in \mathbb{R}^{3\times k} \text{ when } k > 2.$$

- The matrix of the "true" body vectors is:

$$U = R^{\mathrm{T}} E = [u_1 \; u_2 \; u_1 \times u_2] \text{ when } k = 2, \text{ and}$$
$$U = R^{\mathrm{T}} E = [u_1 \; u_2 \; ... u_k] \in \mathbb{R}^{3\times k} \text{ when } k > 2.$$

Attitude Determination on SO(3)

Wahba's function as a Morse function on SO(3)

- Find an estimated attitude $\hat{R} \in \mathrm{SO}(3)$ such that a weighted sum (with wieght $w_i > 0$) of the squared norms of the vector errors $s_i = e_i - \hat{R}u_i^m$ is minimized (Wahba's problem):

$$\text{Minimize}_{\hat{R}}\ \mathcal{U}^0 = \frac{1}{2}\sum_{i=1}^{k} w_i (e_i - \hat{R}u_i^m)^{\mathrm{T}}(e_i - \hat{R}u_i^m), \tag{2.1}$$

- Equivalently, minimize $\mathcal{U}^0(\hat{R}, U^m) = \frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W\rangle$ with respect to $\hat{R}$ for positive diagonal $W = \mathrm{diag}(w_i)$, $i = 1, \ldots, k$.
  - Here $\langle\cdot,\cdot\rangle$ denotes the trace inner product on the (linear) space of $m \times n$ matrices, defined as: $\langle A, B\rangle = \mathrm{tr}(A^{\mathrm{T}}B)$, $A, B \in \mathbb{R}^{m\times n}$.
  - That $\mathcal{U}^0(\hat{R}, U^m)$ is a Morse function on SO(3) with four non-degenerate critical points, was first pointed out by Sanyal (2006 ACC).
- Static attitude determination on SO(3) as solution to (2.1) was obtained in 2006 by:
  - the singular value decomposition (SVD) of $L = EW(U^m)^{\mathrm{T}}$ by Markley;
  - and the QR decomposition of L by Sanyal.
  - Both methods yield the same solution, but the SVD is numerically more efficient than the QR method.

Attitude Determination on SO(3)

In the absence of measurement errors, $U^m = U = R^{\mathrm{T}}E$. Define $Q = R\widehat{R}^{\mathrm{T}}$ as the attitude estimation error and let $W = W^{\mathrm{T}} > 0$ be a positive definite matrix (not necessarily diagonal). The critical points of this generalization of Wahba's cost function are given by the following lemma.

Lemma 0

Let $E \in \mathbb{R}^{3\times k}$ be as defined earlier and let rank$(E) = 3$. Let the gain matrix W of the generalized Wahba cost function be designed as,

$$W = E^T(EE^T)^{-1}K(EE^T)^{-1}E, \tag{2.2}$$

where $K = \text{diag}([k_1, k_2, k_3])$ and $k_1 > k_2 > k_3 > 0$. Then,

$$\mathcal{U} = \frac{1}{2}\left\langle E - \widehat{R}U^m, (E - \widehat{R}U^m)W \right\rangle = \langle K, I - Q\rangle\,, \tag{2.3}$$

is a Morse function on SO(3) *whose disjoint non-degenerate critical points are given by,*

$$Q \in \{I, \text{diag}([-1,-1,1]), \text{diag}([1,-1,-1]), \text{diag}([-1,1,-1])\} \tag{2.4}$$

and $\mathcal{U}$ has a global minimum at $Q = I$.

If $k_1 = k_2$ or $k_2 = k_3$ or $k_1 = k_2 = k_3$, then the cost function $\mathcal{U}$ in eq. (2.3) is a Morse-Bott function, which is a generalization of a Morse function with a closed submanifold of critical points with non-degenerate Hessians normal to this submanifold. Morse-Bott functions on SO(3) have also been used to design attitude observers, e.g., Lageman, Trumpf, and Mahony (2010).

Background: Attitude and Pose Observers on SO(3) and SE(3)

- Attitude estimation using unit quaternions on $\mathbb{S}^3$: using generalizations of extended Kalman filtering: MEKF, AEKF, unscented EKFs.
- Attitude and pose estimation on SO(3) and SE(3):
  - Near-optimal minimum energy filtering: Mortensen's method (1968), solved for SO(3) and SE(3) upto second order in Zamani et al. (2013).
  - Complementary filters, e.g., Mahony et al. (2008), Bonnabel et al. (2009), Hua et al. (2017), Berkane & Tayebi (2018)).
  - Gradient-like observers (e.g., Lageman et al. (2010), Vasconcelos et al. (2010), Hua et al. (2011))
  - Stochastic estimators on SO(3) and SE(3) including invariant Kalman filtering (e.g., Barrau et al. (2017), T. Lee (2018)).
  - "Semi-stochastic methods" like particle filtering, e.g., Bohn & Sanyal (2012).
  - Hybrid schemes (e.g., Wu et al. (2016), Wang & Tayebi (2017)).
- The variational attitude estimator (VAE) in continuous and *discrete* time (Izadi & Sanyal (2014), Izadi et. al. (2015).
- The variational pose estimator (VPE) in continuous and *discrete* time (Izadi & Sanyal (2015, 2016)).

Nonlinearly Stable Observer Designs on SO(3)

- Most of the observers designed on SO(3) are not guaranteed to be nonlinearly stable:
  - Near-optimal and gradient descent observers can become unstable in the presence of measurement noise or when sampled in discrete time,
  - Stochastic estimators can become unstable if the assumed statistics of measurement noise is wrong.
- Further, many of these attitude observers (e.g., complementary filters) do not estimate angular velocity, and assume rapid measurements of angular velocity are available to update attitude estimates.
- Both continuous-time and discrete-time versions of the variational attitude estimator are almost globally asymptotically stable (actually, exponentially stable).

Variational Estimation

This concept is based on constructing a fictitious "mechanical energy" like quantity from estimation errors of the states of a mechanical system. This "energy" is then dissipated in the form of a dissipative "mechanical system" obtained using the Lagrange-d'Alembert principle of variational mechanics. The "equations of motion" of this "mechanical system" give rise to the observer equations.

Lagrangian of the state estimate errors

- The "potential energy" is (the generalization of) Wahba's cost function: $\mathcal{U}(\hat{R}, U^m) = \Phi\Big(\frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W\rangle\Big)$.
  - $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a class-$\mathcal{K}$ function, so $\mathcal{U}$ and $\mathcal{U}^0$ have the same non-degenerate critical points on SO(3) with the same indices.
- The "kinetic energy" is $\mathcal{T}(\hat{\Omega}, \Omega^m, \hat{\beta}) = \frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^{\mathrm{T}}(\Omega^m - \hat{\Omega} - \hat{\beta})$.
  - $m > 0$ is an observer gain, Ω^m is the measured angular velocity, where $\Omega^m = \Omega + \nu + \beta$.
  - $\nu \in \mathbb{R}^3$ and $\beta \in \mathbb{R}^3$ are the additive noise and angular velocity bias vectors, respectively.
  - $\hat{\beta} \in \mathbb{R}^3$ is the estimated angular velocity bias.
- The Lagrangian is $\mathcal{L}(\hat{R}, U^m, \hat{\Omega}, \Omega^m, \hat{\beta}) = \mathcal{T}(\hat{\Omega}, \Omega^m, \hat{\beta}) - \mathcal{U}(\hat{R}, U^m)$ $= \frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^{\mathrm{T}}(\Omega^m - \hat{\Omega} - \hat{\beta}) - \Phi\Big(\frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W\rangle\Big)$.
- The "total energy" function is $\mathcal{E}(\hat{R}, U^m, \hat{\Omega}, \Omega^m, \hat{\beta}) = \mathcal{T}(\hat{\Omega}, \Omega^m, \hat{\beta}) + \mathcal{U}(\hat{R}, U^m)$ $= \frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^{\mathrm{T}}(\Omega^m - \hat{\Omega} - \hat{\beta}) + \Phi\Big(\frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W\rangle\Big)$.

Applying Lagrange-d'Alembert principle

- The action functional constructed from the Lagrangian is:

$$\mathcal{S}(\mathcal{L}(\hat{R}, U^m, \hat{\Omega}, \Omega^m)) = \int_{t_0}^{T} \left(\mathcal{T}(\hat{\Omega}, \Omega^m, \hat{\beta}) - \mathcal{U}(\hat{R}, U^m)\right) \mathrm{d}s$$

$$= \int_{t_0}^{T} \left\{ \frac{m}{2}(\Omega^m - \hat{\Omega} - \hat{\beta})^{\mathrm{T}}(\Omega^m - \hat{\Omega} - \hat{\beta}) - \Phi\left(\frac{1}{2}\langle E - \hat{R}U^m, (E - \hat{R}U^m)W\rangle\right) \right\} \mathrm{d}s.$$

- The angular velocity measurement residual is: $\omega := \Omega^m - \hat{\Omega} - \hat{\beta}$.

- Apply the Lagrange-d'Alembert principle to the action functional $\mathcal{S}(\mathcal{L}(\hat{R}, U^m, \hat{\Omega}, \Omega^m))$, in the presence of a dissipation $\tau_D = D\omega$, where $D = D^{\mathrm{T}} > 0$. This leads to the VAE.

The (continuous time) Variational Attitude Estimator

Theorem 1 (Izadi et al. 2016)

Applying the Lagrange-d'Alembert principle to the action functional $\mathcal{S}(\mathcal{L}(\hat{R}, U^m, \hat{\Omega}, \Omega^m))$ defined earlier, with a dissipative "torque" $\tau_D = D\omega$ where $D = D^T > 0$, leads to the following observer equations for the attitude and angular velocity states:

$$\begin{cases} \dot{\hat{R}} = \hat{R}\hat{\Omega}^\times = \hat{R}(\Omega^m - \omega - \hat{\beta})^\times, \\ m\dot{\omega} = -m\hat{\Omega} \times \omega + \Phi'\big(\mathcal{U}^0(\hat{R}, U^m)\big) S_L(\hat{R}) - D\omega, \\ \hat{\Omega} = \Omega^m - \omega - \hat{\beta}, \end{cases}$$

where $\hat{R}(t_0) = \hat{R}_0$, $\omega(t_0) = \omega_0 = \Omega_0^m - \hat{\Omega}_0$, $S_L(\hat{R}) = \mathrm{vex}\big(L^T\hat{R} - \hat{R}^T L\big) \in \mathbb{R}^3$, $L = EW(U^m)^T$, and $\mathrm{vex}(\cdot) : \mathfrak{so}(3) \to \mathbb{R}^3$ is the inverse of $(\cdot)^\times$.

Bias estimation for the VAE

Proposition 1

Let β be a constant bias in angular velocity measurements. Then, in the absence of measurement noise, the variational attitude estimator of Theorem 1 along with the following update equation for the bias estimate:

$$\dot{\hat{\beta}} = \Phi'\big(\mathcal{U}^0(\hat{R}, U^m)\big)P^{-1}S_L(\hat{R}),$$

is almost globally asymptotically stable (AGAS) for $P \in \mathbb{R}^{3\times 3}$ positive definite.

The following Lyapunov function is used to prove the convergence of estimation errors:

$$V(Q, \omega, \tilde{\beta}) = \frac{m}{2}\omega^{\mathrm{T}}\omega + \Phi\big(\langle I - Q, K\rangle\big) + \frac{1}{2}\tilde{\beta}^{\mathrm{T}}P\tilde{\beta}.$$

Note that $\beta = 0$ (unbiased angular velocity) is a special case of the above result. In the presence of bounded measurement noise and errors, the VAE is shown to be Lyapunov stable with bounded estimation errors (Izadi et al., 2016).

Discrete-time Lagrangian

- Discrete-time stable versions of the VAE are required for computer simulations and implementations, including onboard computer (microprocessor) implementations on autonomous vehicles.
  - Observers that are stable in continuous time may not retain their stability when sampled in discrete time, as shown in Hamrah et al (2018, 2019).
- The "potential energy" in the measurement residual for attitude is discretized as:

$$\mathcal{U}(\hat{R}_i, U_i^m) = \Phi\Big(\mathcal{U}^0(\hat{R}_i, U_i^m)\Big) = \Phi\Big(\frac{1}{2}\langle E_i - \hat{R}_i U_i^m, (E_i - \hat{R}_i U_i^m) W_i \rangle\Big), \; i \in \mathbb{W},$$

  where W_i is a positive definite matrix of weight factors for the measured directions at time t_i.
  - W_i may be time-varying and designed according to Lemma 1.
- The "kinetic energy" in the angular velocity measurement residual is discretized as:

$$\mathcal{T}(\hat{\Omega}_i, \Omega_i^m) = \frac{m}{2}(\Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i)^{\mathrm{T}}(\Omega_i^m - \hat{\Omega}_i - \hat{\beta}_i),$$

  where m is a positive scalar.

Discrete-time VAE

The discrete-time Lagrangian so obtained is $\mathcal{L}(\hat{R}_i, U_i^m, \hat{\Omega}_i, \Omega_i^m, \hat{\beta}_i) = \mathcal{T}(\hat{\Omega}_i, \Omega_i^m, \hat{\beta}_i) - \mathcal{U}(\hat{R}_i, U_i^m)$. From this, we can obtain the discrete action sum $\mathcal{S}(\mathcal{L}(\hat{R}_i, U_i^m, \hat{\Omega}_i, \Omega_i^m, \hat{\beta}_i)) = \sum_{i=0}^{N} h\mathcal{L}(\hat{R}_i, U_i^m, \hat{\Omega}_i, \Omega_i^m, \hat{\beta}_i)$, where h is a fixed time step size.

Theorem 2 (Izadi et al., 2016)

A discrete-time variational attitude estimator obtained by applying the discrete Lagrange-d'Alembert principle to the above discrete action sum is:

$$\begin{aligned}
\hat{R}_{i+1} &= \hat{R}_i \exp\left(h(\Omega_i^m - \omega_i - \hat{\beta}_i)^\times\right),\\
\hat{\beta}_{i+1} &= \hat{\beta}_i + h\Phi'\left(\mathcal{U}^0(\hat{R}_i, U_i^m)\right)P^{-1}S_{L_i}(\hat{R}_i),\\
\hat{\Omega}_i &= \Omega_i^m - \omega_i - \hat{\beta}_i,\\
m\omega_{i+1} &= \exp(-h\hat{\Omega}_{i+1}^\times)\Big\{(mI_{3\times3} - hD)\omega_i\\
&\qquad + h\Phi'\left(\mathcal{U}^0(\hat{R}_{i+1}, U_{i+1}^m)\right)S_{L_{i+1}}(\hat{R}_{i+1})\Big\},
\end{aligned}$$

where $S_{L_i}(\hat{R}_i) = \text{vex}(L_i^T\hat{R}_i - \hat{R}_i^T L_i) \in \mathbb{R}^3$, $L_i = E_i W_i (U_i^m)^T \in \mathbb{R}^{3\times3}$, *$P$ is as defined by Proposition 1, and* $(\hat{R}_0, \hat{\Omega}_0) \in \mathrm{SO}(3) \times \mathbb{R}^3$ *are initial state estimates.*

Note: the $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ map is evaluated using Rodrigues' rotation formula. The above discrete time observer is in the form of a Lie group variational integrator (Leok & Marsden (2003), Sanyal (2004), etc.).

Numerical Simulation Parameters

- The rigid body moment of inertia (unknown to VAE) is $J_v = \text{diag}([2.56 \ \ 3.01 \ \ 2.98]^{\text{T}})$ kg.m^2.
- External torque (unknown to VAE) applied to the rigid body $\varphi(t) = [0 \quad 0.028\sin(2.7t - \frac{\pi}{7}) \quad 0]^{\text{T}}$ N.m.
- Initial attitude and angular velocity are

$$R_0 = \exp\left(\left(\frac{\pi}{4} \times [\frac{3}{7} \quad \frac{6}{7} \quad \frac{2}{7}]^{\text{T}}\right)^{\times}\right)$$

$$\text{and } \Omega_0 = \frac{\pi}{60} \times [-2.1 \quad 1.2 \quad -1.1]^{\text{T}} \text{ rad/s.}$$

- A set of at least two inertial sensors and rate gyros measuring the angular velocity vector, are assumed to be onboard the body.
- Time interval of $T = 40$ s, and time stepsize of $h = 0.01$ s.

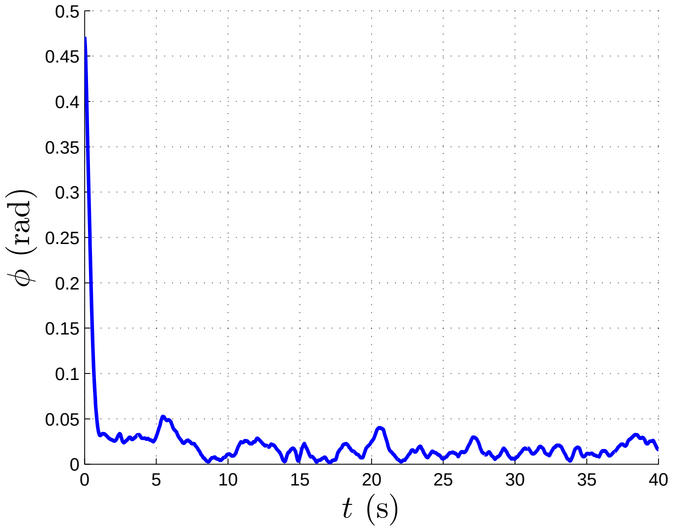
- Three sinusoidal noises with frequencies 1, 10, and 100 Hz and amplitudes of up to 2.4° are added to direction measurements.
- Two sinusoidal noises of 20 and 100 Hz are also added to the angular velocity, with magnitudes of up to 0.97°/s.
- Constant bias in gyro reading $\beta = [-0.01 \quad -0.005 \quad 0.02]^{\mathrm{T}}$ rad/s.
- The estimator Gains $m = 5$, $P = 2 \times 10^3 I$, and $D = \mathrm{diag}\big([17.4 \quad 18.85 \quad 20.3]^{\mathrm{T}}\big)$.
- The initial state and bias estimates

$$\hat{R}_0 = \exp\left(\left(\frac{\pi}{2.5} \times [\frac{3}{7} \quad \frac{6}{7} \quad \frac{2}{7}]^{\mathrm{T}}\right)^{\times}\right),$$

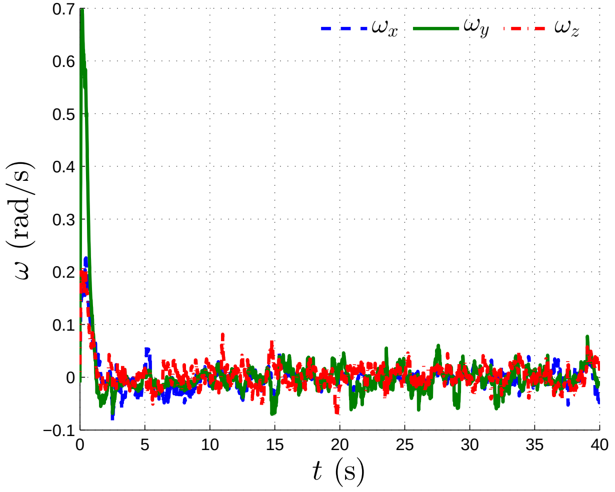
$$\hat{\Omega}_0 = [-0.26 \quad 0.1725 \quad -0.2446]^{\mathrm{T}} \text{ rad/s},$$

$$\text{and } \hat{\beta}_0 = [0 \quad -0.01 \quad 0.01]^{\mathrm{T}} \text{ rad/s}.$$

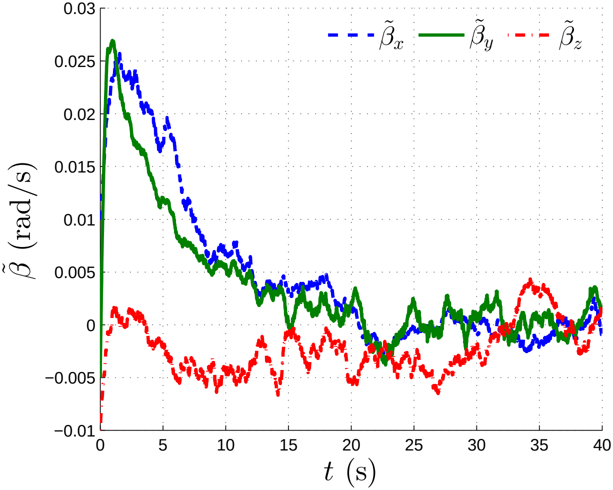
Plot of attitude estimation error (principal angle)



Plot of angular velocity estimation error



Plot of bias estimation error



Numerical Simulations and Comparisons

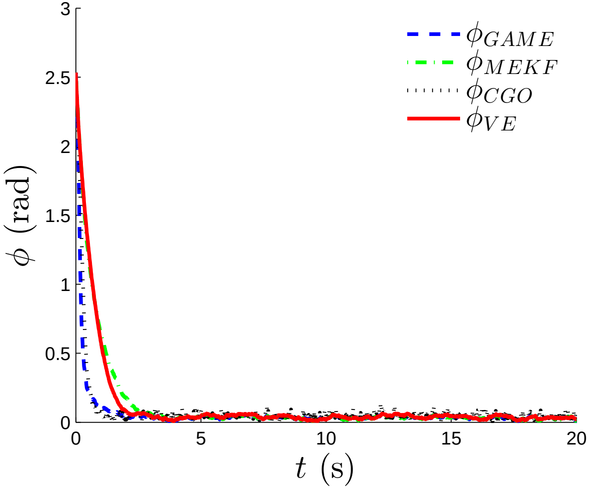
- The variational attitude estimator is compared with three other state-of-the-art attitude estimators
- The attitude estimators being compared to are:
  - the geometric approximate minimum-energy (GAME) filter by Zamani et al (2013)
  - the "industry-standard" multiplicative extended Kalman filter (MEKF), e.g., Markley (2003)
  - a constant gain observer (CGO) in the form of a complementary filter, Mahony et al (2008)
- All estimators start with identical initial estimates and estimate error, and have identical measurement noise added to a given attitude profile

- Sampling interval $h = 0.01$s, time duration $T = 20$s.
- $E = I_{3\times 3}$, $\hat{R}_0 = I_{3\times 3}$ and unbiased sensors.
- The initial attitude is selected randomly about the identity with standard deviation in principal angle $\sigma(\phi(R_0)) = 60°$.
- The rigid body has the following angular velocity profile:

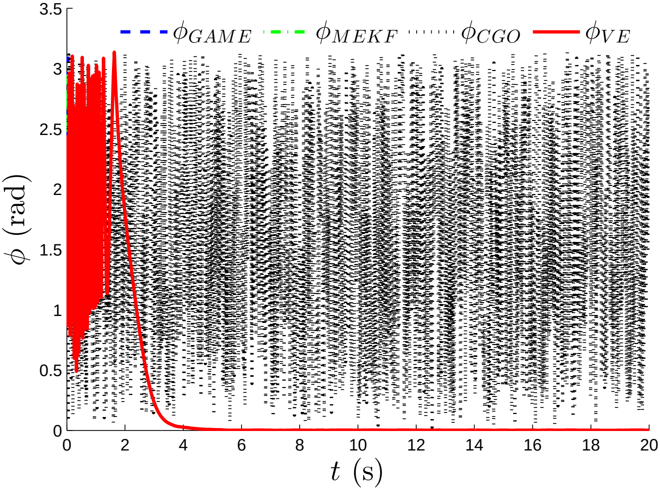
$$\Omega = \begin{bmatrix} \sin(\frac{2\pi}{15}t) \\ -\sin(\frac{2\pi}{18}t + \frac{\pi}{20}) \\ \cos(\frac{2\pi}{17}t) \end{bmatrix}$$

- $D = \text{diag}\big([1.8 \quad 1.95 \quad 2.1]\big)$ N.s and $W = \text{diag}\big([1.67 \quad 1.11 \quad 0.56]\big)$.
- Two cases are simulated: (1) with high noise levels for which all the filter gains are designed; (2) with low noise levels for which none of the gains are designed.

CASE 1: HIGH NOISE LEVELS



Case 2: Low Noise Levels, with Filter Gains as Before



CASE 2: LOW NOISE LEVELS, ZOOM IN AND RUN TIME

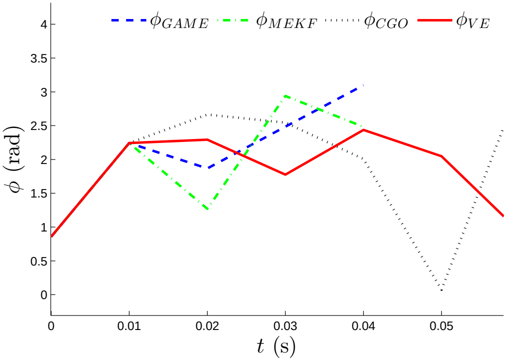
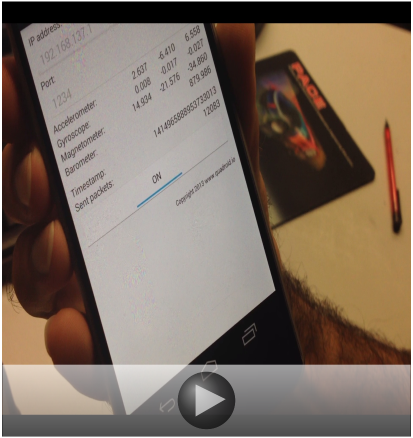


Table: Run-time of different estimators

Estimator	GAME	MEKF	CGO	Var. Est.
Run-Time	0.6864 s	0.6240 s	0.4304 s	0.1716 s

Video of Experiment



Frame Definitions

Inertial landmarks on O with coordinate frame O as observed from vehicle S with coordinate frame S, using vision and/or LiDAR sensor measurements. This information can be used for instantaneous determination of the pose or dynamic pose estimation.

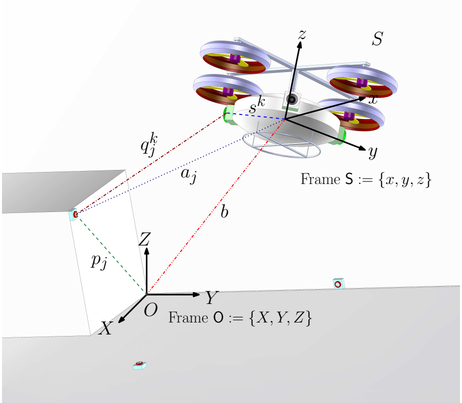


Figure: Coordinate frame definitions for pose estimation

Pose Measurement Model

- Referring to Fig. ??, vector measurements $p_j = R(q_j^k + s^k) + b = Ra_j + b,\ j \in \mathcal{I}(t)$
- For LiDAR range measurements, $a_j^m = (q_j^k)^m + s^k = (\varrho_j^k)^m u^k + s^k,\ j \in \mathcal{I}(t)$.
- The mean of these vectors satisfies $\bar{a}^m = R^{\mathrm{T}}(\bar{p} - b) + \bar{\varsigma}$, where $\bar{p} = \frac{1}{j}\sum_{j=1}^{j} p_j$, $\bar{a}^m = \frac{1}{j}\sum_{j=1}^{j} a_j^m$.
- $d_j = Rl_j \Rightarrow D = RL$, where $D = [d_1 \ \cdots \ d_n]$, $L = [l_1 \ \cdots \ l_n] \in \mathbb{R}^{3\times n}$, $d_i = p_\lambda - p_\ell$, $l_j = a_\lambda - a_\ell$, for $\lambda, \ell \in \mathcal{I}(t)$, $\lambda \neq \ell$.
- $L^m = R^{\mathrm{T}} D + L$

Velocity Measurements and State Estimation errors

- Velocities Ω and ν are measured directly using inertial sensors, radar, and/or lidar; or by filtering position vector measurements in the body-fixed coordinate frame.
- The estimated pose and its kinematics

$$\hat{g} = \begin{bmatrix} \hat{R} & \hat{b} \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \quad \dot{\hat{g}} = \hat{g}\hat{\xi}^\vee.$$

- The pose estimation error

$$h = g\hat{g}^{-1} = \begin{bmatrix} Q & b - Q\hat{b} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Q & x \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3),$$

where $Q = R\hat{R}^\mathrm{T}$ and $x = b - Q\hat{b}$.

- In the case of perfect measurements, $\dot{h} = h\varphi^\vee$, where $\varphi(\hat{g}, \xi^m, \hat{\xi}) = \begin{bmatrix} \omega \\ v \end{bmatrix} = \mathrm{Ad}_{\hat{g}}(\xi^m - \hat{\xi})$, where $\mathrm{Ad}_{\mathfrak{g}} = \begin{bmatrix} \mathcal{R} & 0 \\ \mathfrak{b}^\times \mathcal{R} & \mathcal{R} \end{bmatrix}$ for $\mathfrak{g} = \begin{bmatrix} \mathcal{R} & \mathfrak{b} \\ 0 & 1 \end{bmatrix}$.

Lagrangian from Measurement Residuals

- Trace inner product $\langle A_1, A_2 \rangle := \text{tr}(A_1^{\text{T}} A_2)$.
- Attitude potential energy function $\mathcal{U}_r^0(\hat{g}, L^m, D) = \frac{1}{2}\langle D - \hat{R}L^m, (D - \hat{R}L^m)W \rangle$.
- Translational potential energy function $\mathcal{U}_t(\hat{g}, \bar{a}^m, \bar{p}) = \frac{1}{2}\kappa y^{\text{T}} y = \frac{1}{2}\kappa \|\bar{p} - \hat{R}\bar{a}^m - \hat{b}\|^2$, where $y \equiv y(\hat{g}, \bar{a}^m, \bar{p}) = \bar{p} - \hat{R}\bar{a}^m - \hat{b}$ and κ is a positive scalar.
- Total potential energy function $\mathcal{U}(\hat{g}, L^m, D, \bar{a}^m, \bar{p}) = \Phi\big(\mathcal{U}_r^0(\hat{g}, L^m, D)\big) + \mathcal{U}_t(\hat{g}, \bar{a}^m, \bar{p})$.
- Kinetic energy-like function $\mathcal{T}\Big(\varphi(\hat{g}, \xi^m, \hat{\xi})\Big) = \frac{1}{2}\varphi(\hat{g}, \xi^m, \hat{\xi})^{\text{T}}\varphi(\hat{g}, \xi^m, \hat{\xi})$.

- The Lagrangian $\mathcal{L}(\hat{g}, L^m, D, \bar{a}^m, \bar{p}, \varphi) = \mathcal{T}(\varphi) - \mathcal{U}(\hat{g}, L^m, D, \bar{a}^m, \bar{p})$.
- The action functional $\mathcal{S}\big(\mathcal{L}(\hat{g}, L^m, D, \bar{a}^m, \bar{p}, \varphi)\big) = \int_{t_0}^{T} \mathcal{L}(\hat{g}, L^m, D, \bar{a}^m, \bar{p}, \varphi)\mathrm{d}t$
- Rayleigh dissipation term $\mathbb{D}\varphi$ where $\mathbb{D} = \mathbb{D}^{\mathrm{T}} \in \mathbb{R}^{6\times 6} \succ 0$.
- Apply the Lagrange-d'Alembert Principle: $\delta_{h,\varphi}\mathcal{S}\big(\mathcal{L}(h, D, \bar{p}, \varphi)\big) = \int_{t_0}^{T} \eta^{\mathrm{T}}\mathbb{D}\varphi\mathrm{d}t$.

The Variational Pose Estimator

Theorem 2

The nonlinear variational estimator for pose and velocities is:

$$\begin{cases} \dot{\varphi} & = \mathrm{ad}^*_{\varphi}\varphi - Z(\hat{g}, L^m, D, \bar{a}^m, \bar{p}) - \mathbb{D}\varphi, \\ \dot{\hat{\xi}} & = \xi^m - \mathrm{Ad}_{\hat{g}^{-1}}\varphi, \\ \dot{\hat{g}} & = \hat{g}(\hat{\xi})^\vee, \end{cases}$$

where $\mathrm{ad}^*_\zeta = (\mathrm{ad}_\zeta(\,))^\mathrm{T}, \mathrm{ad}_\zeta(=) \begin{bmatrix} w^\times & 0 \\ v^\times & w^\times \end{bmatrix}$ for $\zeta = \begin{bmatrix} w \\ v \end{bmatrix} \in \mathbb{R}^6$, and

$$Z(\hat{g}, L^m, D, \bar{a}^m, \bar{p}) = \begin{bmatrix} \Phi'\Big(\mathcal{U}^0_r(\hat{g}, L^m, D)\Big) S_\Gamma(\hat{R}) + \kappa \bar{p}^\times y \\ \kappa y \end{bmatrix},$$

where $S_\Gamma(\hat{R}) = \mathrm{vex}\big(DW(L^m)^\mathrm{T}\hat{R}^\mathrm{T} - \hat{R}L^m W D^\mathrm{T}\big)$.

Note that some velocities may not be measured directly, in which case they are substituted by filtered position vectors.

Discrete-time VPE Equations

First-order discrete-time VPE

$$
\begin{aligned}
(J\omega_i)^\times &= \frac{1}{\Delta t}(F_i\mathcal{J} - \mathcal{J}F_i^{\mathrm{T}}),\\
(M + \Delta t\mathbb{D}_t)\upsilon_{i+1} &= F_i^{\mathrm{T}}M\upsilon_i + \Delta t\kappa(\hat{b}_{i+1} + \hat{R}_{i+1}\bar{a}^m_{i+1} - \bar{p}_{i+1}),\\
(J + \Delta t\mathbb{D}_r)\omega_{i+1} &= F_i^{\mathrm{T}}J\omega_i + \Delta tM\upsilon_{i+1} \times \upsilon_{i+1}\\
&\quad + \Delta t\kappa\bar{p}^\times_{i+1}(\hat{b}_{i+1} + \hat{R}_{i+1}\bar{a}^m_{i+1})\\
&\quad - \Delta t\Phi'\big(\mathcal{U}^0_r(\hat{g}_{i+1}, L^m_{i+1}, D_{i+1})\big)S_{\Gamma_{i+1}}(\hat{R}_{i+1}),\\
\hat{\xi}_i &= \xi^m_i - \mathrm{Ad}_{\hat{g}_i^{-1}}\varphi_i,\\
\hat{g}_{i+1} &= \hat{g}_i\exp(\Delta t\hat{\xi}^\vee_i),
\end{aligned}
$$

where $F_i \in \mathrm{SO}(3)$, $\big(\hat{g}(t_0), \hat{\xi}(t_0)\big) = (\hat{g}_0, \hat{\xi}_0)$, $\varphi_i = [\omega_i^{\mathrm{T}}\ \upsilon_i^{\mathrm{T}}]^{\mathrm{T}}$, and $S_{\Gamma_i}(\hat{R}_i)$ is the value of $S_\Gamma(\hat{R})$ at time t_i.

Note that this is obtained as a Lie group variational integrator (LGVI) scheme (Leok & Marsden (2003), Sanyal (2004), etc.,) on $\mathrm{TSE}(3) \simeq \mathrm{SE}(3) \times \mathbb{R}^6$, which is computationally efficient and respects the geometry of SE(3).

Numerical Simulation Parameters

- The vehicle mass and moment of inertia are taken to be $m_v = 420$ g and $J_v = [51.2\ \ 60.2\ \ 59.6]^{\mathrm{T}}$ g.m^2, respectively.
- $\phi_v(t) = 10^{-3}[10\cos(0.1t)\ \ 2\sin(0.2t)\ \ -2\sin(0.5t)]^{\mathrm{T}}$ N and $\tau_v(t) = 10^{-6}\phi_v(t)$ N.m
- The vehicle's initial attitude and position are:

$$R_0 = \mathrm{expm}_{\mathrm{SO(3)}}\left(\left(\frac{\pi}{28} \times [3\ \ 6\ \ 2]^{\mathrm{T}}\right)^{\times}\right),$$

$$\text{and } b_0 = [2.5\ 0.5\ -3]^{\mathrm{T}} \text{ m}.$$

- Its initial angular and translational velocity are:

$$\Omega_0 = [0.2\ \ -0.05\ \ 0.1]^{\mathrm{T}} \text{ rad/s},$$

$$\text{and } \nu_0 = [-0.05\ \ 0.15\ \ 0.03]^{\mathrm{T}} \text{ m/s}.$$

- Time interval of $T = 150$ s, and time stepsize of $h = 0.02$ s.
- The estimator gains are

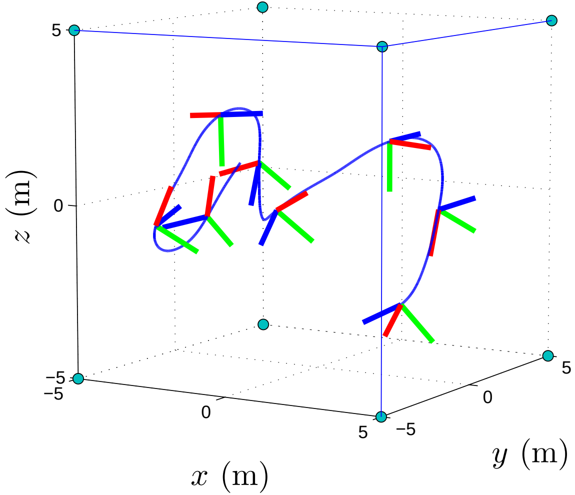
$$\begin{aligned} J &= \text{diag}\big([0.9 \;\; 0.6 \;\; 0.3]\big), \\ M &= \text{diag}\big([0.0608 \;\; 0.0486 \;\; 0.0365]\big), \\ \mathbb{D}_r &= \text{diag}\big([2.7\; 2.2\; 1.5]\big), \mathbb{D}_t = \text{diag}\big([0.1 \;\; 0.12 \;\; 0.14]\big). \end{aligned}$$

- Initial state estimates are

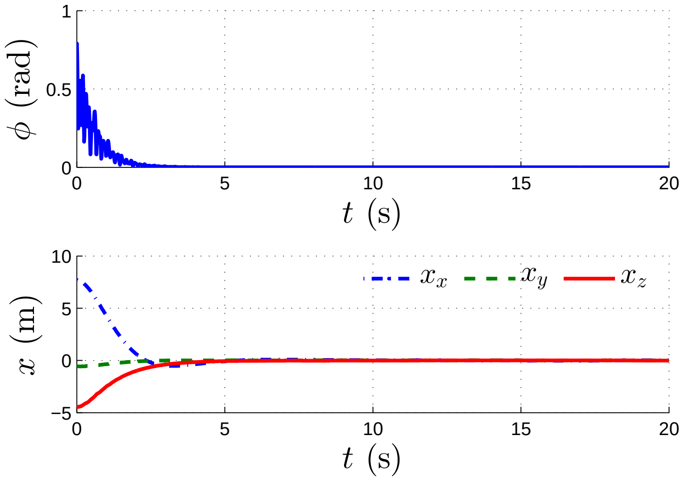
$$\hat{g}_0 = I, \quad \hat{\Omega}_0 = [0.1 \;\; 0.45 \;\; 0.05]^{\mathrm{T}} \text{ rad/s},$$

$$\text{and } \hat{\nu}_0 = [2.05 \;\; 0.64 \;\; 1.29]^{\mathrm{T}} \text{ m/s}.$$

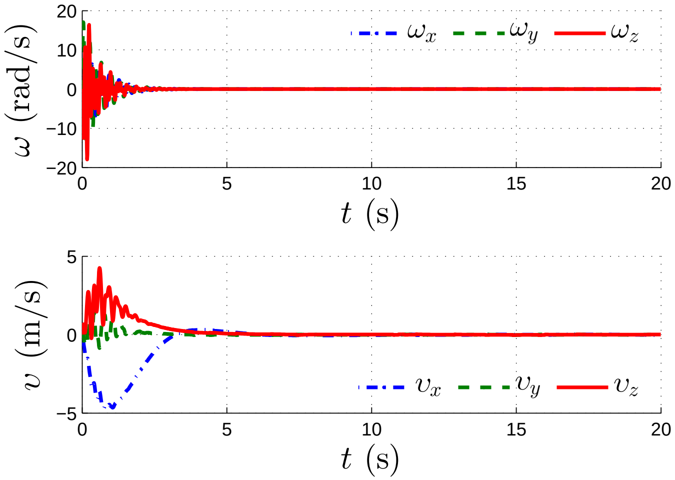
Position and attitude trajectory of the simulated vehicle



Attitude and position estimation error



Angular and translational velocity estimation error



Conclusions and Ongoing Work

- The VAE and VPE designs in continuous and discrete time give almost globally asymptotically stable (AGAS) observers for attitude and pose estimation respectively.
- The stability of these schemes have been rigorously proven, and the discrete time VAE was compared with that of a sampled continuous observer designs on SO(3) (often implemented on a computer using quaternions). Numerical results show that the discrete-time VAE is stable and more reliable for computer implementation when measurement noise properties are unknown and computational efficiency is a measure.
- Hölder-continuous finite-time stable (FTS) versions of the VAE and VPE have been or are being developed in continuous time and in discrete time.
  - FTS version of the VAE in continuous time has already been reported (Sanyal et al. (ECC 2019), Wang et al. (CDC 2019)).
  - Discrete-time FTS versions of the VAE and VPE are being developed; the theory of Hölder-continuous FTS systems in discrete time was recently developed in late 2019.
- Ongoing work is trying to implement the discrete time VPE in indoor flight experiments on a quadrotor UAV to demonstrate its performance in practice.

Challenges Ahead

- The biggest challenge ahead is convincing the research community about the use of *nonlinearly stable* observers on Lie groups for rigid body (and multi-body) systems.
    - In particular, design of *discrete time stable* observers on Lie groups for such systems for (onboard) computer implementation.
- Related challenges:
    - Educating current and future researchers (doctoral students) about the use of Lie group observer designs for rigid body and multi-body systems.
    - Making researchers realize the use of the *softer* aspects of differential geometry and topology in observer and controller design.
- Transitioning these results into practice.
    - Through funding from federal agencies (convincing program managers of the usefulness of these schemes).
    - Through industry collaborations.

Thank you!

Questions? Comments?