An Optimal Kalman-Consensus Filter for Distributed Implementation over Dynamic Communication Network

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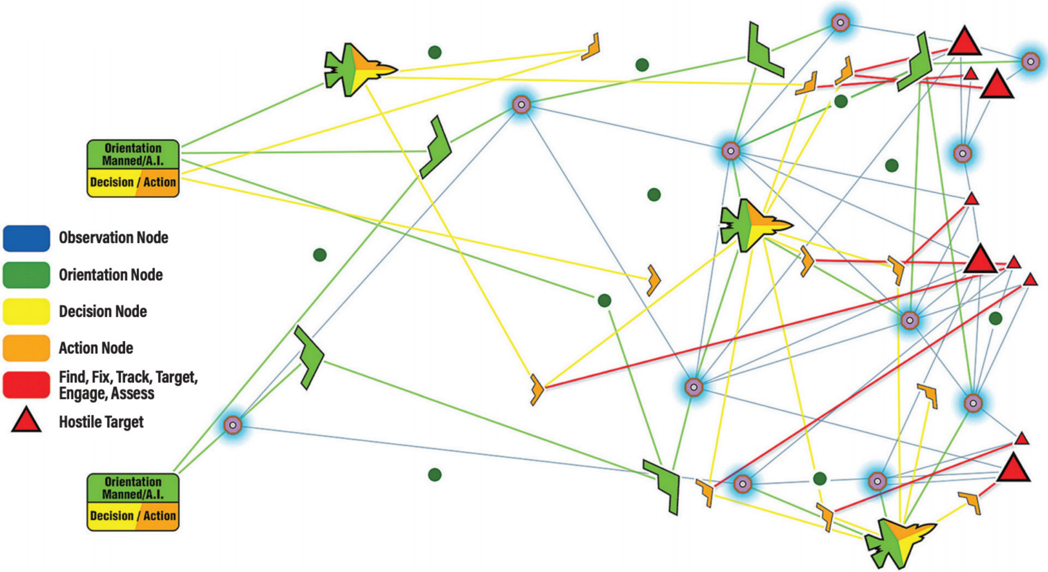
An Application Scenario: Mosaic Warfare



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Target

- Linear stochastic system:

$$x_k = \Phi_{k-1} x_{k-1} + B_{k-1} u_{k-1} + G_{k-1} w_{k-1}$$

  with state $x_k \in \mathbb{R}^n$ and input $u_k \in \mathbb{R}^m$.
- Observation:

$$z_k = H_k x_k + v_k$$

  or, for $i = 1, \cdots, N$,

$$z_{i,k} = H_{i,k} x_k + v_{i,k}$$

- iid stochastic processes:

$$\begin{aligned} E\{w_k\} &= 0, \qquad E\{w_k w_\kappa^T\} = Q_k \delta(k, \kappa) \\ E\{v_k\} &= 0, \qquad E\{v_k v_\kappa^T\} = R_k \delta(k, \kappa), \qquad E\{v_k w_\kappa^T\} = 0; \\ E\{v_{i,k}\} &= 0; \qquad E\{v_{i,k} v_{j,\kappa}^T\} = R_{i,k} \delta(k, \kappa) \delta(i, j), \qquad E\{v_{i,k} w_{i,\kappa}^T\} = 0. \end{aligned}$$

Kalman Filter (KF)

It follows from R. Kalman [4] that, letting

$$P_k^+ = E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\},$$

then $\operatorname{tr}\left(P_k^+\right)$ is minimized by using

- Propagation:

$$\begin{aligned}
\hat{x}_k^- &= \Phi_{k-1}\hat{x}_{k-1}^+ + B_{k-1}u_{k-1} \\
P_k^- &= \Phi_{k-1}P_{k-1}^+\Phi_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T
\end{aligned}$$

- Update:

$$\begin{aligned}
\hat{x}_k^+ &= \hat{x}_k^- + K_k(z_k - H_k\hat{x}_k^-) \\
P_k^+ &= (I - K_kH_k)P_k^-(I - K_kH_k)^T + K_kR_kK_k^T \\
K_k &= P_k^-H_k^T(H_kP_k^-H_k^T + R_k)^{-1}
\end{aligned}$$

Convergence of KF

- The system of $x_k = \Phi_{k-1} x_{k-1}$ and $z_k = H_k x_k$ is observable if

$$\mathcal{O} = \begin{bmatrix} H_k & H_k\Phi_k & H_k\Phi_k^2 & \ldots & H_k\Phi_k^{n-1} \end{bmatrix}^T$$

  is of full rank.

- Under observability, the Discrete-Time Algebraic Riccati Equation (DARE)

$$\begin{aligned} P_k^- = {} & \Phi_{k-1}P_{k-1}^-\Phi_{k-1}^T - \Phi_{k-1}P_{k-1}^-H_{k-1}^T\left(H_{k-1}P_{k-1}^-H_{k-1}^T + R_{k-1}\right)^{-1}H_{k-1}P_{k-1}^-\Phi_{k-1}^T \\ & + G_{k-1}Q_{k-1}G_{k-1}^T \end{aligned}$$

  has at least one P.S.D. solution, and the Kalman filter is asymptotically convergent as $\hat{x}_k^+ \to E\{x_k\}$.

Kalman-Consensus Filter (KCF)

It follows from R. Olfati-Saber [5] that a KCF can be used to cooperatively track a target:

- propagation:

$$\hat{x}_{i,k}^{-} = \Phi_{k-1}\hat{x}_{i,k-1}^{+} + B_{k-1}u_{k-1}$$
$$P_{ij,k}^{-} = \Phi_{k-1}P_{ij,k-1}^{+}\Phi_{k-1}^{T} + G_{k-1}Q_{k-1}G_{k-1}^{T}$$

- Update:

$$\hat{x}_{i,k}^{+} = \hat{x}_{i,k}^{-} + K_{i,k}(z_{i,k} - H_{i,k}\hat{x}_{i,k}^{-}) + M_{i,k}\sum_{j\in\mathcal{N}_i}(\hat{x}_{j,k}^{-} - \hat{x}_{i,k}^{-})$$

$$P_{ij,k}^{+} = F_{i,k}P_{ij,k}^{-}F_{j,k}^{T} - F_{i,k}\sum_{s\in\mathcal{N}_j}(P_{ij,k}^{-} - P_{is,k}^{-})M_{j,k}^{T} - M_{i,k}\sum_{r\in\mathcal{N}_i}(P_{ij,k}^{-} - P_{rj,k}^{-})F_{j,k}^{T}$$
$$+ M_{i,k}\sum_{r\in\mathcal{N}_i}\sum_{s\in\mathcal{N}_j}(P_{ij,k}^{-} - P_{is,k}^{-} - P_{rj,k}^{-} + P_{rs,k}^{-})M_{j,k}^{T} + K_{i,k}R_{ij,k}K_{j,k}^{T}$$

where $F_{i,k} = I - K_{i,k}H_{i,k}$

KCF Gains and Unsolved Issues

- $K_{i,k}$ can be chosen to minimize $\mathrm{tr}\left(\left(\right) P_{ii}\right)^{+}$ as does in KF, that is,

$$K_{i,k} = \left[P^{-}_{ii,k} H^{T}_{i,k} - M_{i,k} \sum_{j \in \mathcal{N}_i} (P^{-}_{ii,k} - P^{-}_{ij,k}) H^{T}_{i,k} \right] (H_{i,k} P^{-}_{ii,k} H^{T}_{i,k} + R_{ii,k})^{-1}$$

- $M_{i,k}$ impacts both consensus and stability, and the choice in [5] is

$$M_{i,k} = \epsilon \frac{P^{-}_{ii,k}}{1 + \|P^{-}_{ii,k}\|}, \quad \epsilon > 0.$$

- Issue #1: KCF requires full knowledge of covariance matrix $P^{-}_{k} = [P^{-}_{ij,k}$. How do we compute P^{-}_{k}? Approximate approaches in literature:
  1. Ignore the cross-covariance - leading to inconsistent estimators
  2. Over estimate the cross-covariances - leading to overly pessimistic estimators
- Issue #2: How to optimize the choice of $M_{i,k}$? How to analyze the resulting convergence?
- Issue #3: Existing KCF requires a constant undirected graph. What happens if the topology is directed and changing?

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Sensing/Communication Topology

- Directed graph $\mathcal{G}_k = (\mathcal{N}, \mathcal{E}_k)$ where

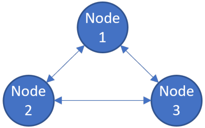
$$\mathcal{N} = \{1, 2, ..., N_m\}, \quad \mathcal{E}_k \subseteq \{(i,j) | i,j \in \mathcal{N} \text{ and } i \neq j\}.$$

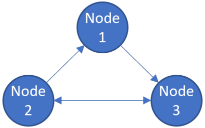
- Laplacian matrix

$$L_k = D_k - A_k$$

where D_k is the degree matrix and A_k is the adjacency matrix.

Examples:





$$L_1 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Distributed Estimation of Laplacian

- If a diagraph if strongly connected, then its Laplacian is irreducible [6] and can be represented by

$$L_k = \sum_{p=1}^{N_m} \lambda_p(L_k) v_p(L_k) v_p^T(L_k)$$

where $\lambda_p(\cdot)$ and $v_p(\cdot)$ are the pth eigenvalues and right eigenvectors, respectively

- Define $\mathcal{Q}_k \triangleq (L_k + I)^{-1}$ as an invertible and irreducible matrix, then

$$\hat{\mathcal{Q}}_{i,k}(l+1) = \hat{\mathcal{Q}}_{i,k}(l) - \frac{1}{|\mathcal{N}_{i,k}|}\mathcal{P}_{i,k}\left(|\mathcal{N}_{i,k}|\hat{\mathcal{Q}}_{i,k}(l) - \sum_{j\in\mathcal{N}_{i,k}} \hat{\mathcal{Q}}_{j,k}(l)\right)$$

can be distributively calculated over $t \in ((k-1)T, kT)T$ [1] where

$$\mathcal{P}_{i,k} = I_n - \frac{1}{[L_k + I]_{i*}[L_k + I]_{i*}^T}[L_k + I]_{i*}^T[L_k + I]_{i*}$$

Network Topology Estimation

- With $\hat{\mathcal{Q}}_{i,k}$ known, the Laplacian can be re-created using

$$\hat{L}_{i,k} = \sum_{p=1}^{N_m} \left[\frac{1}{\lambda_p(\hat{\mathcal{Q}}_{i,k})} - 1 \right] v_p(\hat{\mathcal{Q}}_{i,k}) v_p^T(\hat{\mathcal{Q}}_{i,k})$$

from which the estimated set of in-neighbors of node i can be found via

$$\hat{\mathcal{N}}_{i,k} = \{ j \,|\, [\hat{L}_{i,k}]_{ij} = -1 \}$$

- Convergence in a finite number of steps [1]
- $\hat{\mathcal{N}}_{i,k}$ can be calculated during the higher rate propagation phase

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DOKCF

It follows from Howard and Qu [2] that DOKCF admits time-varying digraphs, enables distributed computation of P_k^-, provides optimal choice of $M_{ij,k}$, and ensures convergence:

- propagation:

$$\hat{x}_{i,k}^- = \Phi_{k-1}\hat{x}_{i,k-1}^+ + B_{k-1}u_{k-1}$$
$$P_{ij,k}^- = \Phi_{k-1}P_{ij,k-1}^+\Phi_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T$$

- update:

$$\hat{x}_{i,k}^+ = \hat{x}_{i,k}^- + K_{i,k}(z_{i,k} - H_{i,k}\hat{x}_{i,k}^-) + \sum_{j\in N_{i,k}} M_{ij,k}(\hat{x}_{j,k}^- - \hat{x}_{i,k}^-)$$

$$P_{ij,k}^+ = F_{i,k}P_{ij,k}^-F_{j,k}^T - F_{i,k}\sum_{s\in N_j}(P_{ij,k}^- - P_{is,k}^-)M_{js,k}^T - \sum_{r\in N_i} M_{ir,k}(P_{ij,k}^- - P_{rj,k}^-)F_{j,k}^T$$
$$+ \sum_{r\in N_i}\sum_{s\in N_j} M_{ir,k}(P_{ij,k}^- - P_{is,k}^- - P_{rj,k}^- + P_{rs,k}^-)M_{js,k}^T + K_{i,k}R_{ij,k}K_{j,k}^T$$

where $F_{i,k} = I - K_{i,k}H_{i,k}$

Optimal Kalman and Consensus Gains

- Minimizing $\operatorname{tr}(P_{ii,k})^+$ with respect to $K_{i,k}$ yields

$$K_{i,k} = \left[P^-_{ii,k} - \sum_{j \in N_{i,k}} M_{ij,k}(P^-_{ii,k} - (P^-_{ij,k})^T)\right] H^T_{i,k} \left(H_{i,k} P^-_{ii,k} H^T_{i,k} + R_{ii,k}\right)^{-1}$$

- As $R_{jj,k}$ increases, $P^+_{j,k}$ increases while $K_{j,k}$ monotonely decreases, and $\hat{x}_j$ becomes worse. Accordingly, the corresponding weighting matrix in DOKCF should be decreased. That is,

$$M_{ij,k} = \alpha_{i,k} \Phi^T_{k-1} \sqrt{R^{-1}_{i,k} R^{-1}_{j,k}}$$

where $\alpha_{i,k}$ is optimized by again minimizing $\operatorname{tr}(P_{ii,k})^+$:

$$\alpha_{i,k} = -\frac{\operatorname{tr}(\Psi_{i4})}{\operatorname{tr}(\Psi_{i5})}$$

where Ψ_{i4} and Ψ_{i5} are quantities defined in [2]

Passivity-Short System

- Define the system error as $e_{i,k} = E\left\{e^{+}_{i,k}\right\}$ such that

$$e_{i,k} = F_{i,k}\Phi_{k-1}e_{i,k-1} + \nu_{i,k-1}$$

where $F_{i,k} = I - K_{i,k}H_{i,k}$ and

$$\nu_{i,k-1} = \sum_{j\in\mathcal{N}_{i,k}} M_{ij,k}\Phi_{k-1}\left(e^{+}_{j,k-1} - e^{+}_{i,k-1}\right)$$

- If there exists a storage function for a dynamic subsystem defined by $e_{i,k} = \mathcal{F}_i(e_{i,k-1}, \nu_{i,k-1})$ such that

$$\begin{aligned}\Delta V_i &\triangleq V_i(e_{i,k}) - V_i(e_{i,k-1}) \\ &\leq e^{T}_{i,k-1}\nu_{i,k-1} + \frac{1}{2}\epsilon_i\|\nu_{i,k-1}\|^2,\end{aligned}$$

then the subsystem is considered *input-feedforward passivity short (PS)* [7, 3]

Stability for DOKCF under Directed Topology

- Following from cooperative control theory [1] with an appropriate storage function of

$$V_i = \frac{1}{2} e_{i,k}^T e_{i,k}$$

  the DOKCF is shown to be *input-feedforward passivity short (PS)* from input $\nu_{i,k-1}$ to output $e_{i,k}$
- To ensure *input-feedforward passivity short (PS)*,

$$\alpha_{i,k} \leq \overline{\alpha}$$

- The DOKCF is, therefore, concluded to be asymptotically stable such that $e_{i,k} \rightarrow 0$
- Details of the proof can be found in [2]

Distributed Implementation I

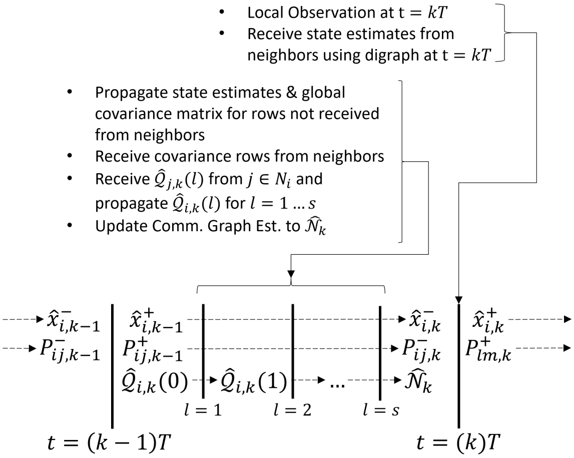
- Recall the global P_k^- needed for optimal and consistent implementation
- Covariance equation do not rely on real-time measurements
- As such, if measurement models $(H_{i,k},\ R_{i,k})$ are known throughout, P_k^- can be calculated by each node
- Simplification can be achieved to reduce N_m^2 covariance computations:
  - Through the covariance definition, P_k^- is symmetry and only one triangle is required $(N_m + \sum_{l=1}^{N_m-1} l)$
  - Rows of P_k^- can be shared between neighbors, neighbors' neighbors, etc.
    - Converges to (N_m), or the same as a complete digraph

Initialization Phase:

- Set all initial conditions (ie. $P^-_{ii,0} = I$ and $P^-_{ij,0} = 0$)
- Share $(H_{i,k},\ R_{i,k})$

Execution Phase:

- Propagate state, covariances, cross-covariances
- Network topology estimation
- Share information with neighbors
- Update state and remaining covariances & cross-covariances



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Setup

- A simple time-varying example was formulated
- Time-varying system model with $u_k = 0$

$$\Phi_k = \begin{bmatrix} 1.0 + 0.025\sin(0.3k) & -0.015 \\ 0.015 & 1.0 + 0.05\sin(0.5k) \end{bmatrix}$$

- The process and measurement noises defined as

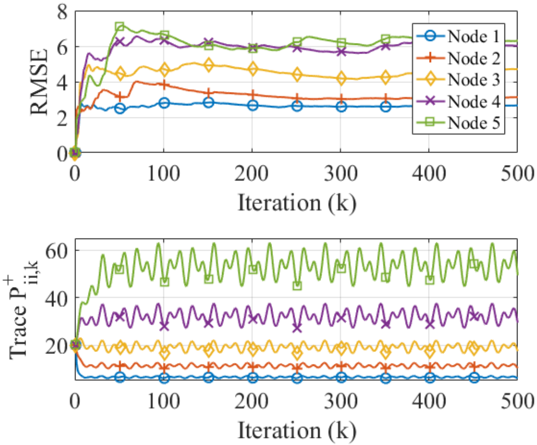
$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_i = 5e^{i} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Measurement noises greater for higher indices

- Root Mean Squared Error used to quantify the accuracy of the estimator

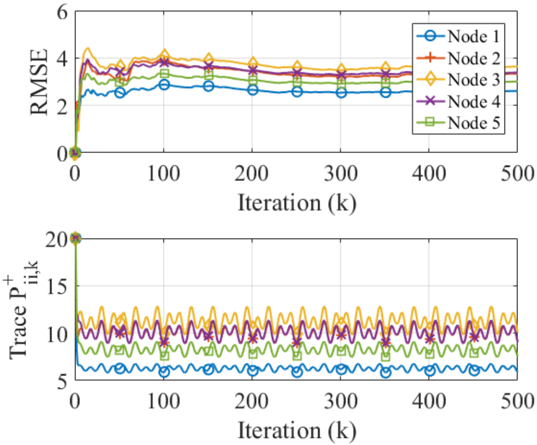
$$RMSE_k = \sqrt{\frac{1}{k}\sum_{s=1}^{k}\left(x_s - \hat{x}_s^+\right)^T\left(x_s - \hat{x}_s^+\right)}$$

- RMSE of the state estimate shows appropriate trend
- Covariances show convergence to a steady-state with oscillation based on time-varying Φ_k
- Results used as a baseline for comparison



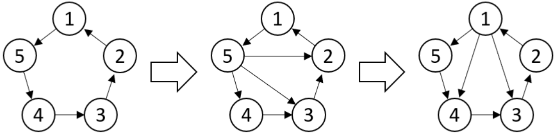
DOKCF with Static Topology

- The DOKCF is simulated using a simple fixed ring network topology
- The RMSE's are better clustered together, as compared to the KF, with:
  - High measurement noise systems decreasing
  - Low measurement noise systems roughly unaffected



DOKCF with Switching Topology I

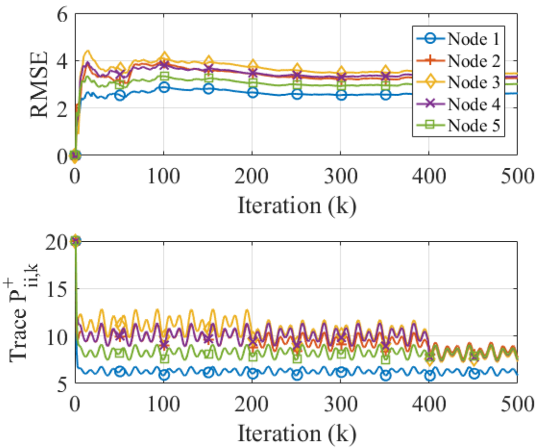
- A switching communication network is introduced with three different topologies
  - Starts with same ring network as before
  - First switch connects system with worst measurement noise, node 5, to nodes 2 and 3
  - Second switch reverts back to ring network but with lowest noise system connected to nodes 3 and 4



- Switching occurs at time steps $k = 200$ and $k = 400$

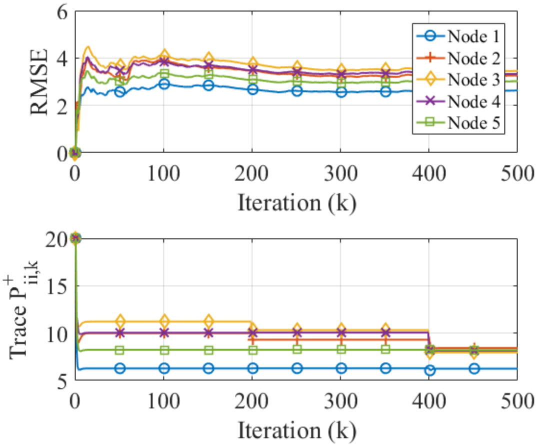
DOKCF with Switching Topology II

- Simulation shows the filters remain stable through the instantaneous topology switches
- RMSE shows consensus has converged and network switching has little effect
- Covariance, however, is impacted based on network topology
  - At $k = 200$, Node 3 shows improvement despite being connected to 5 (worse measurement noise)
  - At $k = 400$, Nodes 2, 3, and 4 decrease due to the distribution of node 1 accurate estimate



DOKCF with Switching Topology for LTI System

- Time-varying perturbations are turned off, forced $k = 0$ in Φ_k, to better see covariance impacts
- Shows filters achieve steady-state covariances between network switches
- Same covariance trend observed at network switches



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Summary and Future Research

- Improvements of DOKCF over KCF:
  - Optimized matrices in weighted consensus for better accuracy
  - Dynamic sensing/communication networks
  - Stability ensured for directed network
  - Distributed computation of P_k^-
- Further research needed to
  - Extend DOKCF to nonlinear systems with nonlinear measurements
  - Stability analysis for switching topology and for nonlinear systems
  - Optimizing network topology

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